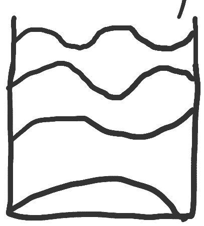
Fermi and Bose distributions

Fermions – in each quantum state there cannot simultaneously be >1 particle
= Fermi–Dirac statistics

Bosons – in each quantum state there may be >1 particles

Consider a system of non-interacting particles in with quantum states labelled by k



← Levels of one particle

In principle, each of those levels may be considered as a separate subsystem

Indeed,

$$\Omega = -T \ln \sum_N e^{\frac{\mu N}{T}} \sum_i e^{-\frac{E_{iN}}{T}}$$

$$N = n_1 + n_2 + \dots$$

$$E_{nN} = n_1 \varepsilon_1 + n_2 \varepsilon_2 + \dots$$

$$\Omega = -T \ln \sum_{\{n\}} e^{\frac{\mu}{T}(n_1+n_2+\dots)} e^{-\frac{\varepsilon_1 n_1 + \varepsilon_2 n_2 + \dots}{T}} =$$

$$= -T \ln \sum_{\{n\}} \prod_k e^{\frac{\mu - \varepsilon_k}{T} n_k} =$$

$$= -T \ln \sum_{\{n\}} \prod_k e$$

$$= -T \ln \prod_k \sum_{n_k} e^{\frac{\mu - \varepsilon_k}{T} n_k} = \sum_k \Omega_k$$

where $\Omega_k = -T \ln \sum_{n_k} e^{\frac{\mu-\varepsilon_k}{T} n_k}$

For fermions $n_k = 0$ or $n_k = 1$

Then $\Omega_k = -T \ln \left(1 + e^{\frac{\mu - \varepsilon_k}{T}}\right)$

The average number of fermions in a state labelled by k is

$$\bar{n}_k \equiv -\frac{\partial \Omega_k}{\partial \mu} = \frac{e^{\frac{\mu-\varepsilon_k}{T}}}{1 + e^{\frac{\mu - \varepsilon_k}{T}}} = \frac{1}{e^{\frac{\varepsilon_k - \mu}{T}} + 1}$$

$$\boxed{f = \frac{1}{e^{\frac{\varepsilon - \mu}{T}} + 1}}$$ – Fermi–Dirac aka Fermi distribution

$0 \leq f \leq 1$

One should use the normalisation condition $\sum_k n_k = N$

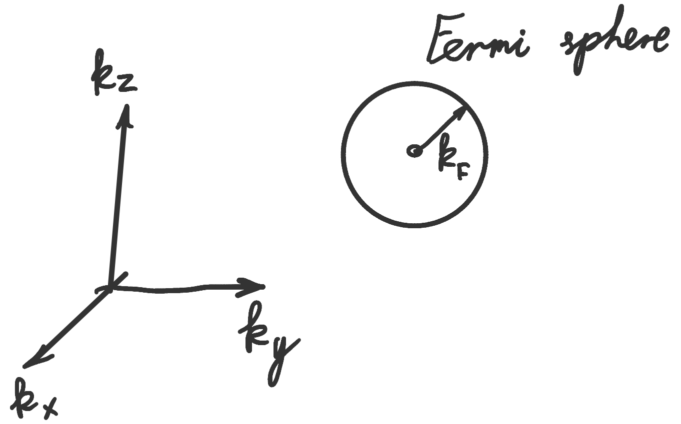
For large temperatures $T \gg \mu$ $\quad n_k \simeq e^{-\frac{\varepsilon_k - \mu}{T}}$

For a given chemical potential μ low energies $\varepsilon \to -\infty$ correspond to $f = 1$ (filled states)

High-energy states with $\varepsilon \to +\infty$ are occupied

High-energy states with ε ...

Example: electrons in a metal:



Bosons

$$\Omega_k = -T \ln \sum_{n_k=0}^{\infty} e^{\frac{\mu-\varepsilon_k}{T} n_k}$$

The geometric series is convergent only for $e^{\frac{\mu-\varepsilon_k}{T}} < 1 \longleftrightarrow \varepsilon_k > \mu$

If $\varepsilon_k = 0$ is the ground state, as it is often convenient to choose,

$$\Omega_k = -T \ln \frac{1}{1-e^{\frac{\mu-\varepsilon_k}{T}}} = T \ln\left(1-e^{\frac{\mu-\varepsilon_k}{T}}\right)$$

$$\bar{n}_k \equiv -\frac{\partial \Omega_k}{\partial \mu} = \frac{1}{e^{\frac{\varepsilon_k-\mu}{T}}-1}$$

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$$\boxed{n_k = \frac{1}{e^{\frac{\varepsilon_k - \mu}{T}} - 1}}$$

– Bose – Einstein aka Bose distrib –

Example: phonons in an oscillator: $\frac{1}{e^{\frac{\hbar\omega}{T}} - 1}$